AP STATISTICS
TOPIC 8: THE BINOMIAL THEOREM

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1. The Binomial Theorem

The Binomial Theorem indicates the coefficients of the polynomial obtained by taking the binomial $(x+a)$ and raising it to the n^{th} power. Here is the statement.

Theorem 1. *Let $x, a \in \mathbb{R}$. Then*

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k}.$$

The first few binomial expansions look like this:

- $(x+a)^0 = 1$
- $(x+a)^1 = x + a$
- $(x+a)^2 = x^2 + 2ax + a^2$
- $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$
- $(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^2x + a^4$
- $(x+a)^5 = ?$

The Binomial Theorem will tell us how to compute the coefficients for $(x+a)^5$; first we need to understand the theorem.

We do not prove this theorem; instead, we attempt to explain each part, and why it is true.

2. Summation Notation

The summation symbol Σ is used to compress notation for adding the first few terms of a sequence. Given real numbers $a_1, a_2, \dots, a_n$, we have

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Typically, the terms a_k are expressed in some formula involving k. For example,

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15,$$

or

$$\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25 = 54.$$

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The number on the bottom tells us the initial value of the index variable k, and the number on the top tells us where to end. For example,

$$\sum_{k=3}^{7} k = 3 + 4 + 5 + 6 + 7 = 25,$$

or

$$\sum_{k=0}^{3} (2k + 1) = 1 + 3 + 5 + 7 = 16.$$

3. Combinatorics

3.1. **Factorials.** The symbol $\binom{n}{k}$ is read "n choose k", and is often called a *binomial coefficient*. It denotes the number of ways to select a subset of k things from a set of n things, where the order of the things selected does not matter.

To figure out how to compute $\binom{n}{k}$, first we figure out the number of ways to arrange a set of n things, or equivalently, the number of ways to select n things from a set of n things, where the order does matter.

Image a bag of marbles. To arrange them, pick out one marble at a time and put them in a row. How many different rows can be made in this way?

Suppose, for example, you have 5 marbles in a bag. First you pick the first one; there are five possible choices. Then you pick the next one; since the first one is missing, there are four possible choices for the second. The first one can be any one of five, and the second any one of the remaining four. Altogether, there are $5 \times 4 = 20$ possible ways to pick the first two marbles. Continuing in this way, there are three choices for the third marble, two choices for the fourth, and only one choice for the last. Altogether, there are $5 \times 4 \times 3 \times 2 \times 1$ possible arrangements of the marbles.

Let n be a nonnegative integer. We define $n!$, read n *factorial*, to be the product of the positive integers less than or equal to n:

$$n! = n \times (n - 1) \times \cdots \times 2 \times 1.$$

The first few of these are

- $0! = 1$ (just live with it)
- $1! = 1$
- $2! = 2$
- $3! = 6$
- $4! = 24$
- $5! = 120$
- $6! = 720$
- $7! = 5040$
- $8! = 322560$

These numbers grow very rapidly as n increases.

By the above reasoning, the number of ways of rearranging n things is $n!$.

3.2. **Permutations.** Next, we wish to find the number of ways of selecting k things from a set of n things, where the order of the selected things does matter. Each way is called a *permutation*.

Again, we have n choices for the first thing, $n - 1$ choices for the second thing, and so on, done to $n - k + 1$ choices for the k^{th} thing selected. Altogether, we have

$n \times (n-1) \times \cdots \times (n-k+1)$ possible ways of selecting k things from a set of n things, where the order does matter.

For example, if we choose 4 things from a set of 9 things, there are

$$9 \times 8 \times 7 \times 6 = 3024$$

possibilities. But there is a shorter way to write this:

$$3024 = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{9!}{5!},$$

where $5 = 9 - 4$.

In general, there are

$$\frac{n!}{(n-k)!} = n \times (n-1) \times \cdots \times (n-1+1)$$

ways to select k things from a set of n things, where the order does matter.

3.3. **Combinations.** Finally, we wish to find the number of ways of selecting k things from a set of n things, where the order does not matter. Each way is called a *combination.*

There are $\frac{n!}{(n-k)!}$ possible ways to pick k marbles out of a bag of n marbles, where the order does matter.

When we pick marbles out of a bag, we pick one at a time (so we can keep a count). There are $k!$ different ways to rearrange these marbles; thus, each set of marbles was counted $k!$ times when we counted the possible permutations. We divide by this to obtain the number of combinations; this is n choose k:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Here are some easy facts about binomial coefficients.

Given n things, there is exactly one way to choose zero things (none of them), and there is exactly one way to choose n things (all of them). Clearly, there are n ways to choose one thing. Finally, choosing k things is the same as leaving $n-k$ things behind, so if you view the situation as choosing what to leave behind, you see that $\binom{n}{k} = \binom{n}{n-k}$. You may use the formula to verify these facts:

- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{k} = \binom{n}{n-k}$

4. Pascal's Triangle

The mathematician Blaise Pascal (French 1623-1662) noticed an inductive formula for computing the binomial coefficients.

Suppose we have already figured out the combinations for a set of n things; from this, can we figure out the combinations for a set of $n+1$ things?

Imagine a bag of n marbles; you have figured out its combinations. You add a new marble, so you now have a bag of $n+1$ marbles. You wish to compute $\binom{n+1}{k}$.

Break this down into two cases: those selections of k marbles which pick the new marble, and those that do not. If you do not pick the new marble, this is the same as not having it there; thus, there are $\binom{n}{k}$ possibilities. On the other hand, if you do pick the new marble, there are now only n marbles in the bag, from which you have to pick $k-1$ more marbles. There are $\binom{n}{k-1}$ possibilities. Since these are the only cases, we see that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Using this formula, we build *Pascal's Triangle*.

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                              1
                          1       1
                      1       2       1
                  1       3       3       1
              1       4       6       4       1
          1       5       10      10      5       1
      1       6       15      20      15      6       1
  1       7       21      35      35      21      7       1
1     8       28      56      70      56      28      8       1
1       9       36      84      126     126     84      36      9       1
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The top row is the zeroth row, the next row is the first row, and so forth. The k^{th} entry in the n^{th} row is $\binom{n}{k}$. A given to is derived from the previous by placing a one to the left of the previous leftmost one, and then adding the two numbers from the previous row directly above a given position. From the formula above, we know that this creates the binomial coefficients.

Note that the sum of the numbers in the n^{th} row is 2^n; why is this?

Consider a set with n elements. To construct a subset, each element is either in or not in the subset; that is, for each element, there are two possibilities, in or not in. Thus there are 2^n choices made to construct a subset, and there are 2^n total subsets. Since are $\binom{n}{k}$ different subsets which contain k elements, we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

5. The Binomial Theorem

We now understand the meaning of the summation sign and the binomial coefficients in the statement of the binomial theorem; it remains to see clearly why it is true. We have

$$(x+a)^n = (x+a)(x+a)\cdots(x+a) \qquad (n \text{ times}).$$

Using the distributive property, expand the expression on the right without "collecting like terms". Each term is of the form $a^k x^{n-k}$ for some k between 0 and n. Where does such a term come from? From each linear factor, we must select either x or a. There are 2^n total terms.

Now combine like terms. For a given k, to get a term of the form $a^k x^{n-k}$, we select a from k of the linear factors, and select x from the others. There are $\binom{n}{k}$ ways to do this, so there are $\binom{n}{k}$ terms of the form $a^k x^{n-k}$. This explains the role of the binomial coefficients in the binomial theorem.

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